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Alexander-Spanier cohomology of a Lie foliation

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Abstract. For a smooth foliated manifold $(M, \mathcal{F})$, the basic and the foliated cohomologies are defined by using the de Rham complex of M. These cohomologies are related with the cohomology of the manifold by a spectral sequence, the de Rham spectral sequence of $\mathcal{F}$ ([17]).

A foliated manifold is an example of space with two topologies, one coarser than other. For these spaces one can define a continuous cohomology ([3]) that, for a foliated manifold, correspond to a continuous foliated cohomology ([13]). In this note we present a construction, for spaces with two topologies, to define continuous basic cohomology and a spectral sequence, similar to the de Rham one for a foliation, to relate continuous basic and foliated cohomology with the cohomology of the space. This construction is based upon the Alexander-Spanier continuous cochains.

Applied to the classifying space of a Lie group, the spectral sequence is isomorphic to the Bott spectral sequence from E_2 on.

For a G-Lie foliation, we give an isomorphism between the E_2 term of the spectral sequence and the *reduced cohomology* of G (in the sense of S-T Hu, [7]) with coefficients in the foliated cohomology of $\mathcal{F}$. This permits us to conclude that both spectral sequences, the de Rham one and the Alexander-Spanier one, are isomorphic for any Riemannian foliation. And, in particular, the topological invariance of these cohomologies.

Complete proofs will appear in [10].

1. Continuous cohomology

First of all, let me recall the classical definitions of cohomology of a group. Denote by

$$(F^*(G, \mathbb{R}),\, \delta)$$

the complex of homogeneous cochains of G, $\varphi \in F^q(G, \mathbb{R})$,

$$\varphi \colon G \times \overset{q+1}{\overbrace{\cdots}} \times G \longrightarrow \mathbb{R},$$

verifying the condition

$$g \cdot \varphi(x_0, x_1, \dots, x_q) = \varphi(gx_0, gx_1, \dots, gx_q)$$

and with differential

$$\delta\varphi(x_0, \dots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \, \varphi(x_0, \dots, \hat{x}_i, \dots, x_{q+1}).$$

Its cohomology $H^*(G)$ is the cohomology of the abstract group G. This definition may be generalized by introducing coefficients: an arbitrary L (i.e., an abelian group in which G acts by group automorphisms). The space of cochains $F^q(G,L)$ is defined as the set of G-maps $G \times \overset{q+1}{\cdots} \times G \to L$.

If we denote by BG_δ the classifying space for principal $G-$bundle, G with the discrete topology, it is

$$H^*(G) \cong H^*(BG_\delta),$$

the algebraic cohomology of G is isomorphic to the topological cohomology of its classifying space as discrete group.

Assume now G is a topological group, and let

$$(F_c^*(G,\mathbb{R}),\, \delta)$$

be the subcomplex of $F^*(G,\mathbb{R})$ of continuous cochains. Its cohomology, $H_c^*(G)$, is the ***continuous cohomology*** of the topological group G. If G is a Lie group, the same cohomology is defined by using smooth cochains. In this case,

$$H_c^*(G) \cong H^*(\mathfrak{g},\mathcal{K})$$

where $\mathfrak{g}$ is the Lie algebra of G, $\mathcal{K}$ a maximal compact subgroup (W. T. van Est [18]).

Finally, one says that φ is a locally trivial or with empty support cochain if there exists a neighbourhood U of e in G such that

$$\varphi(e, x_1, \ldots, x_q) = 0$$

whenever all $x_1, \ldots, x_q$ are in U. Denote by $F_c^*(G,\mathbb{R})_0$ the subcomplex of locally trivial cochains and let

$$\overline{F_c}^*(G,\mathbb{R}) = F_c^*(G,\mathbb{R})/F_c^*(G,\mathbb{R})_0$$

be the quotient complex. Its cohomology, $H_\square^*(G,\mathbb{R})$, is the ***continuous cohomoloy with empty supports***, in the sense of Sze-tse Hu ([7]). In [16] S. Swierczkowski proves that if G is a Lie group, then

$$H_\square^*(G) \cong H^*(\mathfrak{g}).$$

Now let X be a topological space, X' the same set as X, with a finer topology. The ***continuous cohomology*** in this frame,

$$H_c^*(X' \mid X)$$

defined by R. Bott and A. Haefliger, is the cohomology of the cochains on X' which are continuous relative to the topology of X. A cochain is a continuous map

$$c\colon \operatorname{Map}(\triangle^q, X') \longrightarrow \mathbb{R},$$

where $\operatorname{Map}(\triangle^q, X')$ is provided with the topology pulled back from $\operatorname{Map}(\triangle^q, X)_{c.o.}$ via the map induced by $i\colon X' \to X$.

Let G be a topological group, G_δ the same group with the discrete topology. Then

$$H_c^*(BG_\delta \mid BG) \cong H_c^*(G).$$

This isomorphism gives the relation between Bott and Haefliger's concept and the continuous cohomology of groups.

2. Spectral sequence associated to a space with two topologies

Let X be a topological space, X' the same set as X, with a finer topology. Let U be an open set in X. A map

$$\varphi\colon\, U^{p+1} \longrightarrow \mathbb{R}$$

is said to be a *basic Alexander-Spanier p-cochain* in U if it is locally constant when one considers in U^{p+1} the topology induced from X'.

With the obvious restriction maps, the Alexander-Spanier cochains define a presheaf, and this presheaf generates the *sheaf of basic Alexander-Spanier cochains* $AS^*_{(X'|X)}$. With the usual differential

$$\delta\,\varphi(x_0,\dots,x_p) \;=\; \sum_{i=0}^{p} (-1)^i\,\varphi(x_0,\dots,\hat{x}_i,\dots,x_p) \tag{1}$$

we have a resolution

$$AS^0_{(X'|X)} \xrightarrow{\;\delta\;} AS^1_{(X'|X)} \xrightarrow{\;\delta\;} AS^2_{(X'|X)} \xrightarrow{\;\delta\;} \cdots \tag{2}$$

of the constant sheaf $\mathbb{R}_X$ on X.

As a consequence, there is a spectral sequence

$$E_2^{p,q}(X' \mid X) \;=\; H^pH^q(X, AS^*_{(X'|X)}) \;\Rightarrow\; H^{p+q}(X,\mathbb{R})\,. \tag{3}$$

$E_2^{p,0}(X' \mid X)$ is the cohomology of the sections of the sheaves $AS^*_{(X'|X)}$ and will be called the *basic cohomology* of $(X' \mid X)$.

In the cochain definition, we can consider continuous (or smooth, in the appropriate case) rather than arbitrary functions. In this case we say about *continuous* or *differentiable* Alexander-Spanier cohomology. The above constructions for the resolution and spectral sequence work also in the continuous and differentiable cases. In this work we are principally concerned with the continuous and the differentiable cohomologies. If it is necessary, to avoid confusion, we shall write

$${}_dAS_{(X'|X)}\,, \qquad {}_cAS_{(X'|X)}\,, \qquad {}_\infty AS_{(X'|X)}\,,$$

for the discrete (arbitrary functions), continuous or differentiable Alexander-Spanier sheaf.

Remark 2.1. Mostow [13] proves that the continuous cohomology is the cohomology of X with values in the sheaf of continuous functions on X, locally constants in X'. I. e., $H^q(X,\, {}_cAS^0_{(X'|X)})$ is the continuous cohomology of Bott and Haefliger.

Example 2.2. Let $f\colon X \to Y$ be a continuous and closed map, such that each $f^{-1}(y)$ is compact and relatively Hausdorff in X. Take

$$X' = \coprod_{y\in Y} f^{-1}(y)\,,$$

the topological sum of each space $f^{-1}(y)$. Then $E_r^{p,q}(\,X' \mid X\,)$ is the Leray spectral sequence of f.

Example 2.3. (cf. [2]) Let G be a topological group and let BG be its Milnor classifying space. Denote by G_δ the group G with the discrete topology. Then BG_δ is the same set as BG, but with a finer topology. As BG is the semi-simplicial space associated to the nerve NG of G, to compute the spectral sequence

$$E_r(BG_\delta \mid BG) \Rightarrow H(BG) \tag{4}$$

one can use a theorem by Segal ([15, Proposition 5.1]), that asserts

$$E_1(BG_\delta \mid BG) = H^q(BG, AS^p_{(BG_\delta|BG)}) \cong H^q(AS^p(N^*G)),$$

to conclude that this spectral sequence is associated to the double complex

$$AS^p(N^qG)$$

with p as filtrant degree. The differential of the complex is $D = \delta_{1,0} + \delta_{0,1}$, where $\delta_{1,0}\colon AS^p(N^qG) \to AS^{p+1}(N^qG)$ is the differential of Alexander-Spanier cochains and $\delta_{0,1}\colon AS^p(N^qG) \to AS^p(N^{q+1}G)$ is induced by the simplicial structure of NG. For a Lie group this spectral sequence is very close to that considered by Bott and Hochschild, constructed from the Čech-de Rham complex of G,

$$A^p(N^qG).$$

They prove that the E_1 term of this spectral sequence is isomorphic to

$$H_c^{q-p}(G, S^p\mathfrak{g}^*),$$

where $\mathfrak{g}$ is the Lie algebra of G considered as a G-module under the adjoint action, $S^q\mathfrak{g}^*$ denotes the q-th symmetric power, and the subscript c denotes the smooth (or equivalent continuous) cohomology of G with values in $S^q\mathfrak{g}^*$. The Bott spectral sequence is a direct summand of (4), and they are isomorphic from the term E_2 on. In the particular case $q = 0$, as $A^0(NG) = AS^0(NG)$, we have

$$H^*(BG, {}_cAS^0_{(BG_\delta|BG)}) \cong H_c(G).$$

(For this isomorphism, see also [13, Corollary 7.6]).

3. Foliated manifolds

Let $(M, \mathcal{F})$ be a foliated manifold, $M = \cup_{x\in M} L_x$. Denote by $M^{\mathcal{F}}$ the set M with the leaves topology: a basis is formed by the connected components of intersections of open sets of M with leaves. The Alexander-Spanier sheaf and spectral sequence of the foliated manifold will be the associated to $(M^{\mathcal{F}} \mid M)$. We better write $AS_{\mathcal{F}}$ instead of $AS_{(M^{\mathcal{F}}|M)}$ and we use the notations

$$E_2^{p,q}({}_cAS_{\mathcal{F}}) \qquad \text{and} \qquad E_2^{p,q}({}_\infty AS_{\mathcal{F}}),$$

for the continuous and the differentiable cases, respec. $E_2^{s,0}(AS_{\mathcal{F}})$ is the Alexander-Spanier basic cohomology,

$$E_2^{p,0}(AS_{\mathcal{F}}) = H^p_{AS}(M/\mathcal{F}) = H^p(AS^*_{\mathcal{F}}(M), d).$$

For smooth foliations (or C^r-foliations, $r \geq 1$) one can construct the *de Rham spectral sequence* of $\mathcal{F}$. Let $(A^*(M), d)$ be the de Rham complex of M. A smooth form η is said to be *basic* if it satisfies:

$$i_Y\,\eta = 0 \ \text{ and } \ i_Y\,d\eta = 0 \tag{5}$$

for all $Y \in \Gamma\mathcal{F}$, the algebra of vector fields tangent to the foliation, where i_Y is the interior product by Y. The basic forms algebra $A^*_{\mathcal{F}}(M)$ is a differential subcomplex of the de Rham complex $A^*(M)$.

The sheaves $A^*_{\mathcal{F}}$ of germs of basic forms define a resolution of $\mathbb{R}_M$, the sheaf of locally constants functions on M,

$$A^0_{\mathcal{F}} \xrightarrow{d} A^1_{\mathcal{F}} \xrightarrow{d} \cdots \xrightarrow{d} A^k_{\mathcal{F}} \longrightarrow 0\,, \tag{6}$$

where d is the exterior derivative and k is the codimension of $\mathcal{F}$. Associated to this resolution, we have the de Rham spectral sequence of $\mathcal{F}$,

$$E_2^{p,q}(A_{\mathcal{F}}) \;=\; H^p(H^q(M, A^*_{\mathcal{F}})) \;\Rightarrow\; H^{p+q}(M, \mathbb{R})\,. \tag{7}$$

$E_2^{p,0}(A_{\mathcal{F}})$ is the basic de Rham cohomology of $\mathcal{F}$. We denote by

$$H^q_{\mathcal{F}} \;=\; H^q(M, A^0_{\mathcal{F}}) \tag{8}$$

the differentiable foliated cohomology of the foliation. Remark that

$${}_\infty AS^0_{\mathcal{F}} = A^0_{\mathcal{F}}\,,$$

the sheaf of smooth functions on M locally constant along the leaves.

There exists a morphism onto of differential sheaves

$$\Lambda :\ {}_\infty AS^*_{\mathcal{F}} \longrightarrow A^*_{\mathcal{F}}\,.$$

If we take for an open set U of M a p-cochain φ given by the product of $p+1$ smooth functions $f_i \colon U \to \mathbb{R}$, $0 \leq i \leq p$,

$$\varphi(x_0, x_1, \ldots, x_p) = f_0(x_0) f_1(x_1) \ldots f_p(x_p)\,,$$

then

$$\Lambda(\varphi) \;=\; f_0\,df_1 \wedge \ldots \wedge df_p\,.$$

In the general case, for $x \in U$ and $Z_1, \ldots, Z_p \in T_xM$,

$$\Lambda(\varphi)_x(Z_1, \ldots, Z_p) \;=\; \frac{1}{p!} \sum_{\tau \in \mathcal{S}_p} \operatorname{sgn}(\tau) \frac{\partial}{\partial \varepsilon_1} \cdots \frac{\partial}{\partial \varepsilon_p} \varphi(x, \exp_x \varepsilon_1 Z_{\tau(1)}, \ldots, \exp_x \varepsilon_p Z_{\tau(p)})\,|_{\varepsilon_i = 0}\ . \tag{9}$$

This map Λ defines a spectral sequence morphism

$$\Lambda_r^{p,q} :\ E_r^{p,q}({}_\infty AS_{\mathcal{F}}) \longrightarrow E_r^{p,q}(A_{\mathcal{F}})$$

that converges to an isomorphism.

Theorem 3.1. *For Riemannian foliations $E_2^{p,q}(AS_{\mathcal{F}})$ and $E_2^{p,q}(A_{\mathcal{F}})$ are finite dimensional,*

$$E_2^{p,q}({}_cAS_{\mathcal{F}}) \cong E_2^{p,q}({}_\infty AS_{\mathcal{F}})$$

and Λ induces an isomorphism

$$E_2^{p,q}({}_\infty AS_{\mathcal{F}}) \cong E_2^{p,q}(A_{\mathcal{F}})\,.$$

This theorem will be a consequence of the particular case of a Lie foliation.

4. Cohomology of Lie foliations

Let $\mathcal{F}$ be a Lie foliation on M with dense leaves. A suitable description of this structure is the following one: there exists a homomorphism

$$\Pi_1\colon\ \pi_1(M) \longrightarrow G$$

G being a simply connected Lie group, a covering map $\pi\colon \tilde{M} \to M$ associated to the homomorphism, with group of deck transformations Γ, the image of $\pi_1(M)$ by the morphism Π_1, and a locally trivial fibration $\Pi\colon \tilde{M} \to G$, equivariant relative to the action of Γ over $\tilde{M}$ and over G by left product. The fibers of Π are the leaves of the lifting foliation $\tilde{\mathcal{F}}$ and Γ is dense in G.

Proposition 4.1. *Let $\mathcal{F}$ be a Lie foliation on a compact manifold M. The inclusion ${}_\infty AS^*_{\mathcal{F}} \to_c AS^*_{\mathcal{F}}$ induces an isomorphism*

$$E_2({}_\infty AS_{\mathcal{F}}) \cong E_2({}_cAS_{\mathcal{F}})\,.$$

The proof is not very different to that for continuous and differentiable cohomology of a Lie group ([12]). One must construct a "regularisation operator". See [14] and [9] for similar constructions. One could conclude also that $E_2(AS_{\mathcal{F}})$ are finite dimensional, but this will be a consequence of Theorem 3.1 above.

In general there is no isomorphism between the E_1 terms. The torus T^2 foliated by lines of constant irrational slope provides a counter-example: as it is equal to the de Rham foliated cohomology, $H^1(T^2, {}_\infty AS^0_{\mathcal{F}})$ has either infinite dimension or dimension one, depending upon whether the irrational slope is Liouville or diophantine, while $H^1(T^2, {}_cAS^0_{\mathcal{F}})$ has always infinite dimension (cf. [13]).

The G-module structure of $H^q_{\mathcal{F}}$ permits to express the spectral sequences as follows.

Proposition 4.2. *Let $\mathcal{F}$ be a Lie foliation. There exists isomorphisms*

$$E_2^{p,q}(A_{\mathcal{F}}) \cong H^p(\mathfrak{g},\, H^q_{\mathcal{F}})\,,$$

$$E_2^{p,q}({}_\infty AS_{\mathcal{F}}) \cong H^p_{\square}(G, H^q_{\mathcal{F}})\,.$$

For the first isomorphism, see [8]. The second is similar: to get an isomorphism

$$H^q(M,\, {}_\infty AS^p_{\mathcal{F}}) \cong \overline{F}^p_c(G, H^q_{\mathcal{F}})$$

one can use a suitable resolution of ${}_\infty AS^p_{\mathcal{F}}$.

Proposition 4.3. *Let $\mathcal{F}$ be a Lie foliation. There exists an isomorphism*

$$H^p_{\square}(G, H^q_{\mathcal{F}}) \cong H^p(\mathfrak{g}, H^q_{\mathcal{F}}).$$

For $H^q_{\mathcal{F}}$ finite dimensional (a very unusual property), this is a theorem by Swierczkowski ([16]). For the general case, we start with the double complex

$$C^{p,q} = F^p_c(G, \underset{\sim}{A}{}^q_e(G, H^i_{\mathcal{F}})),$$

where $\underset{\sim}{A}{}^q_e(G, H^i_{\mathcal{F}})$ is the space of germs at e of de Rham forms on G with values on $H^i_{\mathcal{F}}$, and $F^p_c(G, \underset{\sim}{A}{}^q_e(G, H^i_{\mathcal{F}}))$ are the continuous homogeneous cochains of G with values on $\underset{\sim}{A}{}^q_e(G, H^i_{\mathcal{F}})$. The two differentiation operators d_1 and d_2 of degree $(1,0)$ and $(0,1)$, respectively, are defined as follows. Let $d_1 = \delta$, the usual differentiation as defined in (1). The operator d_2 is induced by the following one:

$$d_2(\eta) = d_G\eta + \sum_{j=1}^{k} \omega^j \wedge \theta_j \circ \eta,$$

where d_G is the exterior derivative on G, $\omega^1, \ldots, \omega^k$ is a basis of the left invariant 1-forms on G, $\xi_1, \ldots, \xi_k$ a basis of $\mathfrak{g}$ dual of $\{\omega^i\}$ and θ_j stands for the action of $\xi_j \in \mathfrak{g}$ on $H^i_{\mathcal{F}}$. The Proposition follows by a standard spectral sequence argument.

Finally, the structure theorem for Riemannian foliations, by Molino ([11]), permits conclude the Theorem 3.1.

Corollary 4.4. *The de Rham spectral sequence $E_r(A_{\mathcal{F}})$ of a Riemannian foliation is a topological invariant for $r \geq 2$.*

For the basic cohomology, $E_2^{p,0}(A_{\mathcal{F}})$, this result was proved by El Kacimi and Nicolau ([5]). In the general case, it was also proven in ([1]), by a different method.

For arbitrary foliations this is not true. We are considering foliations of codimension 1, without compact leaves. There are well known examples of such foliations in the torus topologically equivalent but with de Rham spectral sequences different. All these foliations, if they are transversally orientable, can be defined by a nonsingular closed 1-form, but to do that it is necessary sometimes to change the smooth structure of M (it is a well known theorem by Sacksteder). This change does not modify the continuous cohomology, but, certainly, it changes the de Rham spectral sequence: $E_2^{1,0}(A_{\mathcal{F}})$ is isomorphic to $\mathbb{R}$, in the new smooth structure, and null in the old. But for a codimension one foliation always

$$E_r^{0,q}({}_\infty AS_{\mathcal{F}}) \cong E_r^{0,q}(A_{\mathcal{F}})$$

for $r \geq 1$, and

$$E_r^{1,0}({}_\infty AS_{\mathcal{F}}) \cong E_r^{1,0}(A_{\mathcal{F}}),$$

for $r \geq 2$.

References

[1] Álvarez López, J.A. and X. Masa, *Morphisms of pseudogroups, foliation maps, and spectral sequence of Riemannian foliations*, Preprint.

[2] Bott, R., *On the Chern-Weil homomorphism and the continuous cohomology of Lie groups*, Advances in Math. **11**(1973), 289–303.

[3] Bott, R., *Some Remarks on Continuous Cohomology*, Proceedings Int. Conf. on Manifolds, Tokyo, 1973, University Tokyo Press (1975) 161–170.

[4] Connes, A., and H Moscovici, *Cyclic cohomology, the Novikov conjecture and hyperbolic groups*, Topology, **29** (1990) 345–388.

[5] El Kacimi, A and M. Nicolau, *On the topological invariance of the basic cohomology*, Math. Ann., **295** (1993) 199–224.

[6] Haefliger, A., "Differentiable cohomology", Cours donn au C.I.M.E., 1976..

[7] Hu, S. T., *Cohomology theory in topological groups*, Michigan Math. J., **1**, (1952), 11–59..

[8] Masa, X., *Cohomology of Lie foliations*, Differential Geometry. Research Notes in Math **32**, 211–214, Pitman, 1985..

[9] Masa, X., *Duality and minimality in riemannian foliations*, Comment. Math. Helv. **67** (1992), 17–27..

[10] Masa, X., *Alexander-Spanier cohomology of Foliated Manifolds*, Preprint.

[11] Molino, P., "Riemannian Foliations", Progress in Mathematics, Birkhäuser, 1988..

[12] Mostow, G. D., *Cohomology of topological groups and Solvmanifolds*, Ann. of Math. **73** (1961) 20–48..

[13] Mostow, M. A., "Continuous cohomology of spaces with two topologies", Memoires of the Amer. Math. Soc. **175**(1976), AMS.

[14] Sarkaria, K.S., *A finiteness theorem for foliated manifolds*, J. Math. Soc. Japan, **30** (1978) 687–696.

[15] Segal, G, *Classifying spaces and spectral sequences*, Inst. Hautes Études Sci., Publ. Math., **34**, (1968) 105–112.

[16] Swierczkowski, S., *Cohomology of group germs and Lie algebras*, Pacific J. of Math. **39** (1971) 471-482.

[17] Vaisman, I., *Variétées riemanniannes feuillétées*, Czechosl. Math. J.**21**(1971), 46–75.

[18] van Est, W.T., *Group cohomology and Lie algebra cohomology in Lie groups, I. II.*, (Kon.) Nederl. Akad. Wetensch. Proc., Ser. A, **56** (1953), 484-504.

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